J. Fluid Mech. (1985), vol. 150, pp. 427–440
Printed in Great Britain

Determining modes and fractal dimension of turbulent flows

By P. CONSTANTIN, C. FOIAS,
Indiana University, Bloomington, IN 47405

O. P. MANLEY
U.S. Department of Energy, Washington DC 20545

AND R. TEMAM
Laboratoire d'Analyse Numérique, CNRS and Université Paris-Sud, 91405 Orsay

(Received 17 January 1984 and in revised form 31 July 1984)

Research on the abstract properties of the Navier–Stokes equations in three dimensions has cast a new light on the time-asymptotic approximate solutions of those equations. Here heuristic arguments, based on the rigorous results of that research, are used to show the intimate relationship between the sufficient number of degrees of freedom describing fluid flow and the bound on the fractal dimension of the Navier–Stokes attractor. In particular it is demonstrated how the conventional estimate of the number of degrees of freedom, based on purely physical and dimensional arguments, can be obtained from the properties of the Navier–Stokes equation. Also the Reynolds-number dependence of the sufficient number of degrees of freedom and of the dimension of the attractor in function space is elucidated.

1. Introduction

In his review paper, Moffatt (1981) drew attention to the increasing interest that mathematicians have shown for fluid-mechanics problems. At the same time he pointed out the existing regrettable communication gap between those mathematicians and other research workers in fluid mechanics. The present paper is an attempt to fill that gap, at least in part, in an important area, viz the application of numerical methods to the solution of Navier–Stokes equations. The ideas presented here are not a new theory of turbulence, but they offer possible new insights. In particular they offer a synthesis of some past and present concepts in fluid mechanics.

Recent, rather abstract, research on the asymptotic properties of Navier–Stokes equations should prove valuable to the furtherance of the use of computers as experimental tools in the study of the dynamics of fluids. Here we shall try to cast those abstract results in a more concrete form. As such they may be more appealing to the practitioners of fluid mechanics, and, we hope, more useful to them. In particular, we show that, with some precisely defined physical entities, there is an intrinsic relationship between the Reynolds number, the number of degrees of freedom describing fluid flow, and the fractal dimension of the Navier–Stokes equations, i.e. the time-asymptotic behaviour of their solutions. We are convinced that, just as a careful, knowledgeable experimentalist would not embark on a course of measurements without proper understanding of the behaviour of his instrumentation, so, ideally, a numerical-simulation expert should not embark on a course of complex computations without a full understanding of the critical mathematical aspects of the problem at hand. However, as rightly pointed out by Moffatt in his review, even without full mathematical rigour one can still make a lot of progress based on one's physical intuition. Nonetheless, complicated nonlinear systems may sometimes behave in a counterintuitive manner; therefore, especially where numerical fluid-flow simulations are concerned, one may be misled and arrive at erroneous conclusions. A recent example of such a pitfall was the observation that the chaotic behaviour of the well-known Lorenz model of Rayleigh–Bénard convection is more or less drastically modified when the model is augmented by the addition of higher-order modes (Franceschini & Tebaldi 1981; Treve & Manley 1982). Thus here it is not only the quantitative nature of the approximation that is affected, but perhaps more significantly the qualitative nature of the approximation is radically changed.

Motivated by those considerations, we have inquired recently into the limited but important question as to the degree of approximation needed to ensure that a numerical solution of the Navier–Stokes equations is at least qualitatively correct. By that we mean that if the exact solution is steady, or periodic, or quasiperiodic, or aperiodic, so respectively is the qualitatively correct approximation. And conversely, when a given approximation is steady, or periodic, or quasiperiodic, or aperiodic, under what conditions can one conclude that the exact solution has the same property? Here we discuss heuristically certain important results applicable to three-dimensional viscous incompressible flows. The corresponding rigorous analysis is presented elsewhere (Constantin *et al.* 1984*a*; Constantin, Foias & Temam 1984*b*), as are the results for the two-dimensional case (Foias *et al.* 1983; Constantin & Foias 1983).

In conventional turbulence theory (Landau & Lifshitz 1959) one estimates the number N of degrees of freedom of a turbulent 3-dimensional flow as

$$N \sim (L_0/L_d)^3, \tag{1}$$

where L_0 is the typical large lengthscale and L_d is the Kolmogorov dissipation length

$$L_d = (\nu^3/\epsilon)^{\frac{1}{4}}, \tag{2}$$

with ϵ the energy-dissipation rate per unit mass and ν the kinematic viscosity. Heretofore such an estimate has been based almost solely on otherwise unsubstantiated dimensional and order-of-magnitude arguments. Here we show how, starting with the Navier–Stokes equations, one can deduce (1) in a rigorous way. Moreover, more significantly, we will show that (1) is in fact only a sufficient upper bound on the number of degrees of freedom needed to approximate turbulent 3-dimensional flows. Loosely speaking, this bound is an estimate of the maximum number of degrees of freedom needed for *any solution* of the Navier–Stokes equations. Of course this bound is a large number, leaving open the possibility that some actual flows are representable by a smaller number of degrees of freedom.

As normally invoked, the conventional estimate of N is statistical in nature, and, strictly speaking, it is applicable to a large ensemble of flows; however, one expects a similar result to be valid in the case of the long-time behaviour of a turbulent fluid driven by body forces. In that case the role of the ensemble averages should be played by appropriate time averages. We will deduce our results by using such averages. To this end for the sake of notational simplicity, we define for any function F the upper bound on the time average of F

$$\overline{\langle F\rangle} = \lim_{t\to\infty} \sup \frac{1}{t}\int_0^t F(t')\,dt'$$

and the lower bound on that average

$$\underline{\langle F\rangle} = \lim_{t\to\infty} \inf \frac{1}{t}\int_0^t F(t')\,dt'.$$

We consider the flow of an incompressible viscous fluid contained in a finite three-dimensional region Ω, with a rigid boundary $\partial\Omega$, governed by the Navier–Stokes equations. Thus the fluid velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ is determined by

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \boldsymbol{f} + \nu\nabla^2\boldsymbol{v}, \tag{3a}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{3b}$$

$$\boldsymbol{v}|_{\partial\Omega} = 0, \tag{3c}$$

$$\boldsymbol{v}(\boldsymbol{r},0) = \boldsymbol{v}_0(\boldsymbol{r}), \tag{3d}$$

where $\boldsymbol{f}$ is the external force per unit mass and p is the pressure divided by density. Alternatively, Ω can be a prism with sides L_0, L_1 and L_2, the boundary condition (3c) being replaced by periodicity conditions and the condition (Temam 1983)

$$\int_\Omega \boldsymbol{v}(\boldsymbol{r},t) = 0. \tag{3e}$$

There are several natural ways in which the intrinsically finite number of degrees of freedom of a three-dimensional flow can be made manifest. Here we discuss in turn two of them, namely the determining modes and the fractal dimension (sometimes called capacity) of the attractors. The concept of determining modes is discussed below in greater detail. By attractors, we mean the time-asymptotic limits of the flow. Their fractal dimensions are dealt with subsequently. Note that the fractal dimension or capacity is that of the attractors as subsets of the function space in which the solution of (3) is represented. It does not bear any transparent relation to the question of whether or not the dissipation regions fill the physical space (Frisch, Sulem & Nelkin 1978).

2. Determining modes

Let $(\boldsymbol{w}_m)_{m-1}^{\infty}$ be a complete orthonormal set of three-dimensional vector-valued eigenfunctions in Ω satisfying the Stokes equations

$$\nabla^2\boldsymbol{w}_m + \nabla g_m = -\lambda_m\,\boldsymbol{w}_m, \quad \nabla\cdot\boldsymbol{w}_m = 0,$$

which, together with the appropriate boundary conditions, determine g_m uniquely. Here the index m stands for subscripts arranged so that the λ_m form an increasing sequence, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$. Now consider the eigenfunction expansion of $\boldsymbol{v}$:

$$\boldsymbol{v}(\boldsymbol{r},t) = \sum_{m-1}^{\infty} c_m(\boldsymbol{v})\,\boldsymbol{w}_m(\boldsymbol{r}),$$

where the $c_m(\boldsymbol{v})$, in general functions of time, are the expansion coefficients with respect to $\boldsymbol{v}$. Consider a finite set of modes, $c_j(\boldsymbol{v})\,\boldsymbol{w}_j, j = 1, 2, \dots, M$, of a solution $\boldsymbol{v}(\boldsymbol{r},t)$

of (3). Let there be any other solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)$, starting from different initial conditions. Now let M be so large that, if the differences between the expansion coefficients $c_m(\boldsymbol{v})$ and $c_m(\boldsymbol{u})$ vanish for long time, i.e. if

$$\lim_{t\to\infty} |c_j(\boldsymbol{v}) - c_j(\boldsymbol{u})| = 0 \quad (j = 1, 2, \ldots, M),$$

then in some sense $\boldsymbol{u}$ and $\boldsymbol{v}$ are equal to one another or, more formally,

$$\lim_{t\to\infty} \int_\Omega |\boldsymbol{u}(\boldsymbol{r}, t) - \boldsymbol{v}(\boldsymbol{r}, t)|^2 = 0.$$

Then such a set of M modes is said to be *determining* (Foias *et al.* 1983).

From the practical point of view, the importance of determining modes lies in that in many respects, such as stability, periodicity, etc. the behaviour of the approximation to $\boldsymbol{v}(\boldsymbol{r}, t)$ consisting only of those modes is the same as that of the true solution. For example, if, for a given problem, the exact solution of (3) is quasi-periodic, then an approximate solution consisting of determining modes is also quasi-periodic. Conversely, if an approximation limited to determining modes is, say, quasi-periodic, then we are assured that so is the exact solution. However, no such assurance obtains if the approximation is of lesser order than that based on the determining modes.

It is necessary to note here that the expansion coefficients $c_m(\boldsymbol{v})$ are not quite equal to those resulting from the use of an eigenfunction expansion truncated at $m = M$, i.e. the mode coefficients of an Mth-order Galerkin approximation, say $c_m^{(M)}(\boldsymbol{v})$ (Treve 1981). However, for practical purposes this difference may be ignored, because, as shown elsewhere, if certain technical criteria are satisfied, then, for M sufficiently large all $c_m^{(M)}(\boldsymbol{v})$ approach $c_m(\boldsymbol{v})$ (Constantin, Foias & Temam 1984*c*).

As is well known, it is not yet certain that regular solutions of (3) exist for all times. However, following common practice we assume that in some sense most of such solutions are regular, and in particular that the vorticity is bounded. We define the maximum dissipation rate ϵ as

$$\epsilon = \nu \langle \overline{\sup_r |\nabla \boldsymbol{u}(\boldsymbol{r}, t)|^2} \rangle \tag{4}$$

where $$|\nabla \boldsymbol{u}(\boldsymbol{r}, t)|^2 = \sum_{i,j=1}^{3} |\partial u_i / \partial x_j|^2.$$

For the present purposes the Kolmogorov length will be defined by (2) with the above value of ϵ.

We now demonstrate the validity of (1), or more precisely we show that a sufficient number M of the determining modes of $\boldsymbol{u}$ is approximately equal to $k_1 N$, where N is given by (1). Here and in the sequel k, k_1, k_2, ... denote dimensionless absolute constants, typically of order unity, depending at most on the shape of the flow field, but not on its size.

On taking the difference between the equations (3*a*) for $\boldsymbol{u}$ and $\boldsymbol{v}$, multiplying the result by $C_j(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{w}_j$, where $C_j(\boldsymbol{u}, \boldsymbol{v}) \equiv c_j(\boldsymbol{u}) - c_j(\boldsymbol{v})$, summing over j from $M+1$ to ∞, and integrating over the volume, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=M+1}^{\infty} [C_j(\boldsymbol{u}, \boldsymbol{v})]^2 + \nu \sum_{j=M+1}^{\infty} [C_j(\boldsymbol{u}, \boldsymbol{v})]^2 \lambda_j$$
$$= \int_\Omega (\boldsymbol{u} - \boldsymbol{v}) \cdot \nabla \boldsymbol{u} \cdot \sum_{j=M+1}^{\infty} C_j(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{w}_j + \int_\Omega (\boldsymbol{v} \cdot \nabla)(\boldsymbol{u} - \boldsymbol{v}) \cdot \sum_{j=M+1}^{\infty} C_j(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{w}_j$$
$$\leqslant k\delta + (\sup_r |\nabla \boldsymbol{u}|) \sum_{j=M+1}^{\infty} [C_j(\boldsymbol{u}, \boldsymbol{v})]^2,$$

where
$$\delta = \sum_{i=1}^{M} \sum_{j=M+1}^{\infty} C_i(\boldsymbol{u},\boldsymbol{v})\, C_j(\boldsymbol{u},\boldsymbol{v}) \int_\Omega (\boldsymbol{w}_i\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w}_i + \boldsymbol{v}\cdot\nabla\boldsymbol{w}_i\cdot\boldsymbol{w}_i),$$

which vanishes as $t\rightarrow\infty$, because, by hypothesis for $1 < i < M$, in that limit $C_i(\boldsymbol{u},\boldsymbol{v})\rightarrow 0$. Or, more transparently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=M+1}^{\infty}[C_j(\boldsymbol{u},\boldsymbol{v})]^2 + 2(\nu\lambda_{M+1} - \sup_r|\nabla\boldsymbol{u}|)\sum_{j=M+1}^{\infty}[C_j(\boldsymbol{u},\boldsymbol{v})]^2 \leqslant \lambda k\delta,$$

whence we easily infer that $[c_1(\boldsymbol{u})\,\boldsymbol{w}_1, \dots, c_M(\boldsymbol{u})\,\boldsymbol{w}_M]$ is a determining set for $\boldsymbol{u}$, provided that

$$\nu\lambda_{M+1} - \overline{\langle\sup_r|\nabla\boldsymbol{u}|\rangle} > 0. \tag{5}$$

But it is known that in three dimensions $\lambda_{M+1} > k_2\lambda_1 M^{\frac{2}{3}}$ (Morse & Feshbach 1953; Metivier 1978), so that (5) holds if

$$\langle\overline{\sup_r|\nabla\boldsymbol{u}|}\rangle < k_2 M^{\frac{2}{3}}\nu\lambda_1.$$

Now we relate the lowest eigenvalue λ_1 to the typical lengthscale of the flow, viz $\lambda_1 \approx 1/L_0^2$. Thus (5) is satisfied if

$$\epsilon^{\frac{1}{2}} \leqslant k_2 M^{\frac{2}{3}}\nu^{\frac{3}{2}}\lambda_1 = \frac{k_2 M^{\frac{2}{3}}\nu^{\frac{3}{2}}}{L_0^2}.$$

That is, it suffices that the number of modes $N \geqslant M \sim k_2^{-\frac{3}{2}}(L_0/L_\mathrm{d})^3$, or to within a constant, we recover the estimate (1). A novel conclusion to be drawn from this result is that the conventional estimate (1) is really an upper bound on the number of modes needed to describe 3-dimensional turbulent flow. In fact the necessary and sufficient number may eventually turn out to be smaller! However, at present it is not at all clear that the number of modes needed is as small as that estimated conventionally (e.g. Landau & Lifshitz 1959) on the basis of the dissipation length determined by the *average* dissipation rate.

The derivation leading up to (5) justifies the common assumption about the physical basis for (1). We see that indeed, when (5) is satisfied, the lengthscale given by λ_{M+1} is sufficiently small to ensure that the energy delivered to the higher-order modes by shear stress at the rate $\sim|\nabla\boldsymbol{u}|$ is effectively damped by molecular viscosity.

3. Variation equations and dimensions of attractors

Before addressing the determination of the fractal dimension or capacity of the attractors of the Navier–Stokes equations, it is useful to indicate how the dimension of an attractor may be found for the case of a finite-dimensional system. A natural way to determine the dimension of a space in which time-asymptotic trajectories are imbedded is to test in some manner the neighbourhood of a given trajectory. In particular, such a test may consist of examining how a small volume element evolves along that trajectory: for, if we find that with a given assigned spatial dimensionality, say D, the small D-volume is shrinking indefinitely, we can conclude that, for $t\rightarrow\infty$, the trajectories cannot fill a D-dimensional volume.

To see how this works consider the so-called variation equations associated with a given system of n differential equations (Pars 1965):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} = \{x_1, x_2, \dots, x_n\},$$

with bounded solutions. Let $\boldsymbol{F}(t)$ be a matrix with elements $\partial f_i[x(t)]/\partial x_j$, with $x_j(t)$ assumed to be known functions of time. Then the variation equations are

$$\dot{\boldsymbol{z}} = \boldsymbol{F}(t)\,\boldsymbol{z}, \quad \boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}. \tag{6}$$

Clearly (6) is a linearization of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ around $\boldsymbol{x}(t)$.

Now consider an n-dimensional infinitesimal volume element

$$Y = |y_1 \wedge y_2 \wedge \ldots \wedge y_n|$$

moving along the trajectory prescribed by $\boldsymbol{x}(t)$, where y_i, $(i = 1, 2, \ldots, n)$ are some solutions of (6), and $\wedge$ signifies the outer vector product. Its evolution is evidently

$$\frac{\dot{Y}}{Y} = \sum_{i=1}^{n} \frac{\partial f_i[x(t)]}{\partial x_i} = \operatorname{Tr} \boldsymbol{F}(t). \tag{7}$$

Therefore if $\operatorname{Tr} \boldsymbol{F} < 0$ the n-volume is shrinking. Moreover, it can be shown then that the dimension of the region in which $\boldsymbol{x}(t)$ resides asymptotically as $t \to \infty$ is less than n. A familiar example of such behaviour is the well-known Lorenz system, which is of third order, but whose asymptotic trajectory lies on a complicated set – the Lorenz attractor – which is in fact of less than three dimensions (Lanford 1976). Of course, less exotic examples are limit cycles (one dimension) and point attractors (zero dimension). We now recall that it has been conjectured that the dimension of an attractor is intimately related to the so-called Lyapounov characteristic numbers, or Lyapounov exponents (Kaplan & Yorke 1979; Russell, Hanson & Ott 1980). This useful relationship has now been proved rigorously (Constantin & Foias 1983; Constantin *et al.* 1984*b*; Babin & Vishik 1983).

To see heuristically how this comes about, and to motivate the subsequent discussion, consider as a concrete example the Lorenz equations, by now a familiar model of deterministic chaos:

$$\left.\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= -\sigma x - y - xz, \\ \dot{z} &= -bz + xy + b(r+\sigma), \end{aligned}\right\} \tag{8}$$

where σ, r and b are positive constants. Here as compared with the original version of the Lorenz system, the variable z of Lorenz has been replaced for convenience by $z + r + \sigma$. The variation equations for (8) are

$$\dot{\mathscr{U}} = \boldsymbol{A}(\boldsymbol{u})\,\mathscr{U}, \tag{9}$$

where $\mathscr{U} = (\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3)$ are three solutions of the linear time-dependent system (9), and $\boldsymbol{u} = (x, y, z)$ is the solution of (8), while

$$\boldsymbol{A}(\boldsymbol{u}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ -\sigma - z & -1 & -x \\ y & x & -b \end{bmatrix}.$$

Evidently $\operatorname{Tr} \boldsymbol{A}(\boldsymbol{u}) = -\sigma - 1 - b < 0$. Therefore the region occupied by the time-asymptotic trajectories of (8) – i.e. the attractor – has a dimension less than three.

Now in order to see whether the dimension of the attractor of (8) is larger (or smaller) than two, we investigate the evolution of a small 2-volume, or a plane surface element made up of say U_1 and U_2, $V \equiv U_1 \wedge U_2$, along the solution u of (8). It is easily verified that

$$\frac{\mathrm{d}|V|^2}{\mathrm{d}t} = 2[\mathrm{Tr}\,A(u)|V|^2 - (V, A(u)\,V)] = 2[\mathrm{Tr}\,A(u) - (\gamma, A(u)\,\gamma)]\,|V|^2, \tag{10}$$

where $\gamma = V/|V|$ is a unit vector in the direction of V, normal to the plane determined by U_1 and U_2. For future reference we define the trace

$$\mathrm{Tr}\,A(u)\,\mathrm{Q}_2(U_1, U_2) = \mathrm{Tr}\,A(u) - (\gamma, A(u)\,\gamma), \tag{11}$$

where $\mathrm{Q}_2(U_1, U_2)$ is the orthogonal projection operator – a projector – onto the plane spanned by U_1 and U_2 (a 2-dimensional space imbedded in a higher-dimensional, 3-dimensional space). Of course, not only the size of V changes with time, but so does its orientation. Should (11) be consistently negative, one can prove that the attractor is of dimension less than two. Therefore, in order to proceed we must evaluate $\mathrm{Tr}\,A\mathrm{Q}_2$. Let $\gamma \equiv (x_0, y_0, z_0)$. It follows from (11) that

$$\mathrm{Tr}\,A\mathrm{Q}_2 = -1 - b - \sigma + \sigma x_0^2 + y_0^2 + b z_0^2 + x_0(z y_0 - y z_0). \tag{12}$$

Let $m = \max\,(1, b, \sigma)$. Then

$$\begin{aligned}\mathrm{Tr}\,A\mathrm{Q}_2 &< -1 - b - \sigma + m + x_0(y_0^2 + z_0^2)^{\frac{1}{2}}\,(y^2+z^2)^{\frac{1}{2}}\\ &< -1 - b - \sigma + m + \tfrac{1}{2}(x_0^2 + y_0^2 + z_0^2)^{\frac{1}{2}}\,(y^2+z^2)^{\frac{1}{2}}\\ &< -1 - b - \sigma + m + \tfrac{1}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}}\\ &= -1 - b - \sigma + m + \tfrac{1}{2}|u(t)|.\end{aligned}$$

Using standard methods in (8), it can be shown readily that on the attractor, with $l = \min\,(1, \sigma)$, $|u(t)| < b(r+\sigma)/2(l(b-1))^{\frac{1}{2}}$. Thus finally there follows

$$|V| \leqslant |U_1(0) \wedge U_2(0)|\exp\left\{\frac{-1-b-\sigma+m+b(r+\sigma)}{4(l(b-1))^{\frac{1}{2}}}\,t\right\}.$$

With the usual parameters for the Lorenz attractor, $(b, r, \sigma) = (\frac{8}{3}, 28, 10)$, we have

$$|V| \leqslant |U_1(0) \wedge U_2(0)|\exp\,(16t), \tag{13}$$

i.e. the surface element V expands at a rate slower than some upper bound, and, in fact, may even be shrinking along some solutions. In order to determine whether the latter condition does, or does not, prevail, we would have to determine the sign of the lower bound for $\mathrm{Tr}\,A\mathrm{Q}_2$. However, our purpose here is not to study the detailed properties of the Lorenz attractor, but rather to clarify the connection between the dimension of the attractor and the Lyapounov exponents. For this purpose, as will be seen below, (13) is sufficient. Therefore we proceed to introduce those exponents in the present context.

Consider again (6), and take $Y = |y_1 \wedge y_2 \wedge \ldots \wedge y_m|$, where y_i are $m < n$ solutions of (6). Further, initially let $y(0)$ be an orthonormal set of vectors, i.e. initially, to within an appropriate constant depending only on m, Y is the volume of an m-dimensional sphere. Then (6) serves to describe the deformation of that sphere into an m-dimensional ellipsoid, and its subsequent evolution. At any time the m-volume of that ellipsoid is proportional to the product of its m semiaxes. Let $\alpha_1(t)$, $\alpha_2(t)$, $\ldots$, $\alpha_m(t)$ denote the time-dependent semiaxes of the ellipsoid. Then

$$\int_0^t \mathrm{Tr}\,F_m(t')\,\mathrm{d}t' = \ln \omega_m(t), \tag{14}$$

where $\omega_m(t) = \alpha_1(t)\,\alpha_2(t)\ldots\alpha_m(t)$, and $\mathrm{Tr}\,\boldsymbol{F}_m(t)$ denotes the trace of the matrix $\boldsymbol{F}(t)$ restricted to the space spanned by the principal axes of the ellipsoid. Now we allow t to become so large that it is reasonably certain that $\boldsymbol{u}(t)$ is in the attractor, or very close to it. Assume that the limit

$$\mu_i = \lim_{t\to\infty} \sup \frac{\ln \alpha_i(t)}{t} \tag{15}$$

exists, then μ_i is the Lyapounov exponent associated with $\alpha_i(t)$ (Oseledec 1968). Note that in the most general case $[\ln \alpha_i(t)]/t$ may not tend to a well-defined limit. However, as long as the α_i are bounded (15) is meaningful. It follows then from (14) that as $t\to\infty$, and the m-volume moves along a trajectory in the attractor of a system of n differential equations, the time integral of the trace of $\boldsymbol{F}_m$ satisfies

$$\frac{1}{t}\int_0^t \mathrm{Tr}\,\boldsymbol{F}_m(t')\,\mathrm{d}t' \sim \sum_{l=1}^{m} \mu_l. \tag{16}$$

On comparing (14) with (16) we see that we have thus related the time evolution of elements of different dimensions in the n-dimensional phase space to the sum of the appropriate Lyapounov exponents. It has been shown elsewhere that for a wide class of dynamical systems, including differential equations of practical interest, the required limits, and hence Lyapounov exponents, exist for most trajectories (Oseledec 1968). As in the discussion following (7), when $\Sigma_{i=1}^{m}\mu_i$ is consistently negative, one can prove that the dimension of the attractor is less than m.

In order to put a finer upper bound on the dimension of the attractor we recall now that there are profound generalizations of the concept of space dimension. In the present context two such generalizations are of interest: first is the Hausdorff dimension, and second is the fractal dimension whose importance in physical sciences was recognized by Mandelbrot (1977). The Hausdorff dimension $d_{\mathrm{H}}(X)$ of an attractor is in fact that referred to in the discussion following (7), (9) and (16). It can be shown that (Constantin *et al.* 1984*b*)

$$d_{\mathrm{H}}(X) < m + \frac{\sum_{i=1}^{m}\mu_i}{|\mu_{m+1}|},$$

provided $\Sigma_{i=1}^{m+1}\mu_i < 0$. We note that, for various technical reasons, here we have introduced the so-called uniform Lyapounov exponents μ_i, defined somewhat differently than those in (15). Specifically, they are determined iteratively as

$$\mu_i = \lim_{t\to\infty}\left\{\ln\frac{\bar{\omega}_i(t)}{\bar{\omega}_{i-1}(t)}\right\}\Big/ t, \tag{17}$$

where $\bar{\omega}_i(t) = \sup \omega_i(t)$, with the supremum taken over all trajectories in the attractor. Obviously, the best value for m is the last integer for which $\Sigma_{i=1}^{m}\mu_i \geqslant 0$, because it yields the smallest bound on $d_{\mathrm{H}}(X)$.

We are now ready to complete the estimate of the upper bound on the Hausdorff dimension of the Lorenz attractor. Thus, with the use of (13), $m = 2$, and $n = 3$, there follows $\mu_1+\mu_2+\mu_3 = -1-b-\sigma = -\frac{41}{3}$, $\mu_1+\mu_2 < 16$, whence

$$d_{\mathrm{H}}(\text{Lorenz}) < 2 + \frac{\mu_1+\mu_2}{\frac{41}{3}+\mu_1+\mu_2} = 2.54.$$

As is well known, here for most trajectories in the attractor, one of the non-negative Lyapounov exponents, say μ_2, is equal to zero. A more precise estimate of the uniform

Lyapounov exponents leads to $d_H(\text{Lorenz}) < 2.409$. It is possible that the use of numerically obtained time averages of Lorenz trajectories could yield an even lower bound on d_H, perhaps closer to the probabilistic dimension estimated to be less than or equal to 2.05 (Farmer, Ott & Yorke 1983).

For infinite-dimensional systems, e.g. partial differential equations such as the Navier–Stokes equations, it is more useful to consider the fractal dimension $d_M(X)$. It is always larger than the Hausdorff dimension, and it could even be infinite when the Hausdorff dimension is finite. Nevertheless, when $d_M(X)$ is finite it serves as an upper bound on the Hausdorff dimension. Besides, the fractal dimension appeals more readily to one's physical intuition. It is based on the number of small volumes needed to fill a region of space (Mandelbrot 1977). More precisely the fractal dimension $d_M(X)$ of an object X is defined as

$$d_M(X) = \lim_{\epsilon\to 0} \sup \frac{\ln n_X(\epsilon)}{\ln(1/\epsilon)},$$

where $n_X(\epsilon)$ is the minimum number of balls of radii $\leqslant \epsilon$ needed to fill, or cover, X. It is clear that $n_x(\epsilon)$ depends on the volume of the 'balls', or in the present case on $\omega_n(t)$. Indeed, a laborious analysis (Constantin & Foias 1983; Constantin *et al.* 1984*b*) shows that

$$d_M(X) \leqslant \max_{1\leqslant l\leqslant m}\left(l+\frac{\sum_{i=1}^{l}\mu_i}{|\bar{\mu}_{m+1}|}\right) \leqslant \max_{1\leqslant l\leqslant m}\left[l+(m+1)\frac{\sum_{i=1}^{l}\mu_i}{\left|\sum_{i=1}^{m+1}\mu_i\right|}\right]. \tag{18}$$

Here again μ_i are the uniform Lyapounov exponents, while

$$\bar{\mu}_i = \lim_{t\to\infty}\frac{\sup[\ln\bar{\alpha}_i(t)]}{t}, \quad \text{and} \quad \bar{\alpha}_i = \sup \alpha_i(t),$$

with the supremum taken again over all the trajectories in the attractor. For practical application the rightmost member of (18) is easiest to evaluate. In (18) one assumes again that $\Sigma_{i=1}^{m+1}\mu_i < 0$.

We now extend these elementary ideas to the attractor of the Navier–Stokes equations.

4. Fractal dimension of the Navier–Stokes attractor

In order to apply the results of §3 it suffices, without any loss of generality, to restrict oneself to the solenoidal (divergence-free) portion of (3). Then the equation of evolution corresponding to (3) is of the form (Temam 1983)

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nu\,\mathrm{A}\boldsymbol{v} + \mathrm{B}(\boldsymbol{v},\boldsymbol{v}) = \boldsymbol{f}, \tag{19}$$

where we have assumed that the force $\boldsymbol{f}$ is a divergence-free vector. Here $-\mathrm{A}$ and $\mathrm{B}(\boldsymbol{p},\boldsymbol{q})$ are the divergence-free parts of the Laplacian ∇^2 and of $\boldsymbol{p}\cdot\nabla\boldsymbol{q}$ respectively, and satisfy their appropriate boundary conditions. Next we linearize (19) about some known solution $\boldsymbol{v}_0(\boldsymbol{r},t)$ by taking the Fréchet derivative of (19). This leads to a variation-like equation for a small departure $\boldsymbol{z}$ from the trajectory $\boldsymbol{v}_0$;

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} + \nu\,\mathrm{A}\boldsymbol{z} + \mathrm{B}(\boldsymbol{v}_0,\boldsymbol{z}) + \mathrm{B}(\boldsymbol{z},\boldsymbol{v}_0) = 0. \tag{20}$$

Note that, because of the orthogonality of $B(\boldsymbol{p}, \boldsymbol{q})$ with respect to $\boldsymbol{q}$, a consequence of incompressibility, there is no contribution to the trace from $B(\boldsymbol{v}_0, \boldsymbol{z})$. In analogy with the considerations of §3, we define $Q_m(U_1, U_2, \ldots, U_m)$ to be the orthogonal projector onto the m-dimensional space (imbedded in an infinite-dimensional phase (function) space). For notational simplicity we designate the finite, mth-rank operators $A_m \equiv AQ_m$, and $B_m \equiv BQ_m$, with the corresponding traces $\mathrm{Tr}\, A_m$ and $\mathrm{Tr}\, B_m$. Further, we let $\mathscr{A}_m = \nu A_m + B_m$. Then as a generalization of (10) we have

$$\frac{\mathrm{d}\ln(|U_1 \wedge \ldots \wedge U_m|^2)}{\mathrm{d}t} = -2\,\mathrm{Tr}\,\mathscr{A}_m(\boldsymbol{v}_0).$$

In the special case of orthogonal, solenoidal vectors U_i, for any linear operator T, we have also

$$\mathrm{Tr}\,\mathrm{T}_m = \sum_{j=1}^{m} \left(\frac{TU_j}{|U_j|}, \frac{U_j}{|U_j|}\right),$$

where $(\cdot,\cdot)$ signifies the inner product. On utilizing (19) to extend the ideas presented in §3 to the case of the Navier–Stokes equations in two and three dimensions, it is possible to prove that there exists a number, N_0, such that if $m \geqslant N_0$ a small m-dimensional volume evolving along the solution of (19) shrinks indefinitely, as $t \to \infty$ (Constantin & Foias 1983; Constantin *et al.* 1984*b*). The key result is that $N_0 \sim N$, as in (1). The proof is very technical and elaborate, and it will not be reproduced here. However, in order to give the reader some idea of what is involved, the two key steps are discussed very briefly below.

In the first step, let W be any attractor such that for all solutions of (19) as $t \to \infty$ the enstrophy is bounded from above, i.e.

$$\sup_{\boldsymbol{v} \in W} \int_{\Omega} |\nabla \boldsymbol{v}|^2 < \infty. \tag{21}$$

Here again the volume ω_m of the m-dimensional ellipsoid evolving along the solution of (19) is

$$\omega_m = \exp\left(-\int_0^t \mathrm{Tr}\,\mathscr{A}_m\,\mathrm{d}t'\right),$$

where we have assumed that initially that volume was an m-dimensional unit sphere. Now, for any $m = 1, 2, \ldots$ define q_m as the smallest possible time-averaged value of the trace of $\mathscr{A}_m$, for all possible projectors Q_m:

$$q_m = \inf_{\boldsymbol{v}} \langle \inf_{Q_m} \mathrm{Tr}\,\mathscr{A}_m \rangle, \tag{22}$$

where $\boldsymbol{v}$ runs over all solutions of (19) with the appropriate boundary condition, and such that the initial conditions $\boldsymbol{v}(\boldsymbol{r}, 0)$ are already in the attractor W. Evidently, q_m is a bound on the sum of the m largest uniform Lyapounov exponents. Then for $q_m > 0$ the analogue of (18) in infinite-dimensional space yields for the fractal dimension

$$d_{\mathrm{M}}(W) \leqslant m[1 + \max_{1 \leqslant l \leqslant m} (-q_l)/q_m]. \tag{23}$$

It is necessary to reiterate that this dimension relates to the attractor in the function space and not to the real space in which the flow takes place.

In the second step, we determine q_l and q_m, followed by the demonstration that $d_M(W) \sim N$ of (1). First, let the upper bound on the energy-dissipation rate be given by

$$\epsilon = \nu \overline{\langle \sup_v \sup_r |\nabla v(r,t)|^2 \rangle} < \infty \tag{24}$$

where, as above, v runs over all solutions of (19) which are in W. We define now L_d by (2) and (24) instead of (2) and (4).

It can be shown that on using (24)

$$|\mathrm{Tr}\, \mathrm{B}_j(\cdot, v(r,t))| < j \sup |\nabla v(r,t)| < j\left(\frac{\epsilon}{\nu}\right)^{\frac{1}{2}} \tag{25}$$

and that

$$\mathrm{Tr}\, \mathrm{A}_j > \lambda_1 + \lambda_2 + \ldots + \lambda_j > k_2 \lambda_1 (1 + 2^{\frac{2}{3}} + \ldots + j^{\frac{2}{3}}). \tag{26}$$

Therefore it follows on substituting (25)–(26) in (22) that for any positive integer j

$$\begin{aligned} q_j &> k_2 \nu\lambda_1 (1 + 2^{\frac{2}{3}} + \ldots + j^{\frac{2}{3}}) - j\left(\frac{\epsilon}{\nu}\right)^{\frac{1}{2}} \\ &> j\nu\lambda_1 \left[\tfrac{3}{5} k_2 j^{\frac{2}{3}} - \left(\frac{L_0}{L_d}\right)^2\right] \\ &= j\nu\lambda_1 (\tfrac{3}{5} k_2 j^{\frac{2}{3}} - N^{\frac{2}{3}}). \end{aligned}$$

For $k_2 m^{\frac{2}{3}} > 2.21 N^{\frac{2}{3}}$ it follows that $q_m > 0$ and

$$\max_{1 \leqslant l \leqslant m} -q_l < 0.4 \nu\lambda_1 k_2^{-\frac{3}{2}} N^{\frac{5}{3}}.$$

On substituting in (23) we finally obtain, for large N,

$$d_M(W) < 4.52 N / k_2^{\frac{3}{2}}. \tag{27}$$

Therefore, apart from a constant of order unity, we find that the bound on the dimension of the attractor of the Navier–Stokes equation is identical with the estimate of the number of modes sufficient for the description of the time-asymptotic behaviour of the solutions to that equation.

5. Relation to the Reynolds number

In our approach, for the case considered in §2, the natural definition of the Reynolds number R is

$$R = \overline{\langle \sup_r |v(r,t)|^2 \rangle}^{\frac{1}{2}} / \nu\lambda_1^{\frac{1}{2}}, \tag{28}$$

while, for the case considered in §4,

$$R = \overline{\langle \sup_v \sup_r |v(r,t)|^2 \rangle}^{\frac{1}{2}} / \nu\lambda_1^{\frac{1}{2}}, \tag{29}$$

where v runs over all solutions of (19) with the appropriate boundary conditions and initial conditions already in the attractor W.

Under these conditions it can be shown that

(i) the number of the determining modes of v is less than $k_3 R^3$, where R is defined by (28);

(ii) the fractal dimension of the attractor W of v is less than $k_4 R^3$, where R is defined by (29).

As an example, referring to (ii), on using methods similar to those leading to (25) and (26), it is possible to show from (22) and (29) that

$$q_m > \tfrac{1}{2}\nu\lambda_1 m(3k_2 m^{\frac{2}{3}} - R^2).$$

From this it follows in turn that the use of arguments leading up to (27) results in

$$d_{\mathrm{M}}(W) \sim R^3. \tag{30}$$

This is a somewhat pessimistic estimate, since it is much higher than the conventionally accepted $N \sim R^{\frac{9}{4}}$ (Landau & Lifshitz 1959). However, it must be noted that, in arriving at (30), we have made no use of any *a priori* knowledge of the spectrum of the flow. The more conventional estimate is based on the assumption that the spectrum of the energy density in the flow is the Kolmogorov spectrum. Indeed if we put in formal terms the arguments adduced by Landau & Lifshitz (1959), we find that for a more general power-law spectrum bounded by wavenumbers $\kappa_0 \ll \kappa_{\mathrm{d}}$, the energy density is given by

$$e = \kappa_0^3 \int |\boldsymbol{u}|^2 \,\mathrm{d}V = \mathrm{const}\,(\epsilon\nu)^{\frac{1}{2}} \int_{\kappa_0}^{\kappa_{\mathrm{d}}} \left(\frac{\kappa_{\mathrm{d}}}{k}\right)^n \mathrm{d}\left(\frac{\kappa}{\kappa_{\mathrm{d}}}\right) \approx \mathrm{const}\,\frac{(\epsilon\nu)^{\frac{1}{2}}}{n-1}\left(\frac{\kappa_{\mathrm{d}}}{\kappa_0}\right)^{n-1},$$

where the averaging volume V is determined by the outer length $\sim 1/\kappa_0$. On defining the Reynolds number in terms of the mean-square value of the velocity of the flow, say $R_{\mathrm{s}} = \kappa_0^{\frac{1}{2}}|\boldsymbol{u}|/\nu$, it follows immediately that

$$\left(\frac{L_0}{L_{\mathrm{d}}}\right)^3 = \left(\frac{\kappa_{\mathrm{d}}}{\kappa_0}\right)^3 \sim R_{\mathrm{s}}^{6/(n+1)},$$

which for $n = \frac{5}{3}$ yields the usual result. This raises then a question of the validity of numerous simulations of fluid flows at relatively low Reynolds numbers, so low in fact that the Kolmogorov spectrum is inapplicable, yet for which the number of degrees of freedom is often predicated on the $R^{\frac{9}{4}}$ rule.

6. Concluding remarks

In this paper we have established the relationship between the existing conventional estimates of the degrees of freedom in Navier–Stokes flow, the rigorous bounds on the number of modes needed to represent such flow in numerical simulations, and the fractal dimension of the Navier–Stokes attractor. We have been able then to relate these bounds to an appropriately defined Reynolds number. We have remarked on the conventional measure of the number of degrees of freedom, $R^{\frac{9}{4}}$, as being a result of *a priori* knowledge of the spectrum of homogeneous isotropic turbulence. Obviously this conventional estimate is inapplicable at low and medium Reynolds numbers. A more conservative estimate, independent of the knowledge of the spectrum, as suggested here, varies as R^3.

An overriding conclusion emerging from the work reported here is that in every sense the Navier–Stokes equation is a closed system. That is, the system is determined by a finite but large number of degrees of freedom. Thus, were one to carry a sufficient number of terms in an approximate solution, the terms beyond become irrelevant. The above conclusion suggests the question as to whether in the statistical description of the solutions to the Navier–Stokes equations it is also required to retain a correspondingly large number of degrees of freedom. Further, are conventional low-order closure attempts potentially as misleading as low-order Galerkin approximations (cf. the Lorenz system)? We do not have as yet unequivocal answers to these questions, but given the lack of universality of low-order closures carried out to date, we rather believe that the answer is in the affirmative, namely, that the number is finite but large. However, as yet we cannot prove this conjecture.

The key to obtaining the results presented in this paper lies in the ability to estimate the values of certain integrals, i.e. certain norms. The limitation on those results, there being only sufficient bounds, but not necessary and sufficient, rests on the limits to our present ability to make such estimates. Even so we have been able to establish useful connections between some old and new concepts in fluid mechanics. As sharper bounds on norms are obtained in the field of nonlinear functional analysis, theoretical and numerical fluid mechanics will inevitably benefit further.

This research was supported in part by USDOE-Office of Basic Energy Sciences under contract DE.ACO2.82ER12049. Discussions with H. D. I. Abarbanel and A. N. Kaufman are gratefully acknowledged. We also thank two of the referees for their constructive comments.

REFERENCES

BABIN, A. V. & VISHIK, M. I. 1983 Attractors of partial differential equations of evolution and estimates of their dimensions. *Usp. Math. Nauk* **38**, 133 (in Russian).

CONSTANTIN, P. & FOIAS, C. 1983 Global Lyapounov exponents, Kaplan–Yorke formulas and the dimension of the attractor for 2D Navier–Stokes equations. *Commun. Pure Applied Maths* (To appear).

CONSTANTIN, P., FOIAS, C., MANLEY, O. P. & TEMAM, R. 1984*a* *C.R. Acad. Sci. Paris* **297**, 599.

CONSTANTIN, P., FOIAS, C. & TEMAM, R. 1984*b* Attractors representing turbulent flows. *Mem. Am. Math. Soc.* (to appear).

CONSTANTIN, P., FOIAS, C. & TEMAM, R. 1984*c* On the large time Galerkin approximation of the Navier–Stokes equations. *SIAM J. Numer. Anal.* **21**, 615.

FARMER, J. D., OTT, E. & YORKE, J. A. 1983 *Physica* **7D**, 153.

FOIAS, C., MANLEY, O. P., TEMAM, R. & TREVE, Y. M. 1983 *Physica* **9D**, 157.

FRANCESCHINI, V. & TEBALDI, C. 1981 *J. Stat. Phys.* **25**, 397.

FRISCH, U., SULEM, P.-L. & NELKIN, M. 1978 *J. Fluid Mech.* **87**, 719.

KAPLAN, J. & YORKE, J. A. 1979 Chaotic behavior of multidimensional difference equations. In *Functional Differential Equations and Approximation of Fixed Points* (ed. H. O. Peitgen & H. O. Walther). Lecture Notes in Mathematics, vol. 730. Springer.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. Addison-Wesley.

LANFORD, O. E. 1976 Qualitative and statistical theory of dissipative systems. *CIME School of Statistical Mechanics*.

LEIB, E. H. 1984 *Commun. Math. Phys.* **92**, 473.

MANDELBROT, B. 1977 *Fractals: Form, Chance and Dimension*. Freeman.

METIVIER, G. 1978 *J. Math. Pures Appl.* **57**, 133.

MOFFATT, H. K. 1981 *J. Fluid Mech.* **106**, 27.

MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics*. McGraw-Hill.

OSELEDEC, V. I. 1968 A multiplicative theorem. Lyapounov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19**, 197.

PARS, L. A. 1965 *A Treatise on Analytical Dynamics*. Heineman.

RUSSELL, D. A., HANSON, J. B. & OTT, E. 1980 *Phys. Rev. Lett.* **45**, 1175.

TEMAM, R. 1983 *Navier–Stokes Equations and Nonlinear Functional Analysis*. NSF/CBMS Reg. Conf. Ser. Appl. Maths, SIAM.

TREVE, Y. M. 1981 *J. Comp. Phys.* **41**, 217.

TREVE, Y. M. & MANLEY, O. P. 1982 *Physica* **4D**, 319.

Note added in proof. The estimate of the fractal dimension of the attractor given here can be improved by means of a remarkable inequality due to Lieb and Thirring. That inequality allows us to replace the averaging actually used in the definition of ϵ, (24), by another one; it yields a larger value for L_{d}, and thus decreases the bound on the fractal dimension of the attractor (Constantin *et al.* 1984*b*; Lieb 1984). However, the significance of this improved averaging is not intuitively obvious on purely physical grounds.